International Journal of Theoretical Physics, Vol. 34, No. 1, 1995

Nonclassical Analysis for the Zakharov–Kuznetsov Equation

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Received March 28, 1994

To provide information on the integrability of the Zakharov–Kuznetsov (ZK) equation, we obtain, by a generalization of the direct method introduced by Clarkson and Kruskal, some new similarity reductions of PDEs and ODEs of the ZK equation which governs nonlinear ion-acoustic waves in a magnetized plasma. A symmetry group explanation is given by the nonclassical method of Bluman and Cole.

1. INTRODUCTION

The nonlinear development of ion-acoustic waves in a magnetized plasma under the restrictions of small wave amplitude, weak dispersion, and strong magnetic fields is described by the Zakharov–Kuznetsov (ZK) equation (Zakharov–Kuznetsov, 1974)

$$Q^{(1)} = u_t + uu_x + u_{xxx} + u_{xyy} = 0 \tag{1}$$

Infeld and Frycz (1989) obtained the solitary-wave solutions of (1). Only four polynomial conservation laws have been given (Shivamoggi, 1989, 1990; Infeld, 1985). Shivamoggi (1990) applied the method of Weiss *et al.* in specialized form to investigate the integrability of the ZK equation (1) via the Painlevé property. But up to now, an inverse scattering transformation for (1) has not been constructed and one does not have much more evidence to confirm its integrability. We need to study the ZK equation (1) further.

In Section 2 we use a generalization of the direct method to reduce (1) to partial differential equations with two independent variables and find new reductions. In Section 3 we give the symmetry group-theoretic expla-

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0020-7748/95/0100-0099$07.50/0

nation of the results in Section 2 by showing that these reductions may also be obtained using the nonclassical method. In Section 4 we discuss the reductions of (1) to ordinary differential equations by using a generalization of the direct method. Section 5 is a summary and discussion of our results.

2. REDUCTIONS TO PDE

In this section we seek reductions of (1) in the special form

$$u(x, y, t) = \alpha(x, y, t) + \beta(x, y, t)w(\zeta(x, y, t), \tau(x, t, y)) \tag{2}$$

Substituting (2) into (1) and demanding that the results be a partial differential equation for $w(\zeta, \tau)$ imposes conditions on α, β, ζ, τ, and their derivatives, the solution of which yields the desired reductions. In the determination of α, β, ζ and τ, we have a certain freedom.

Case 2.1. $\zeta_x \neq 0$. Similar to Clarkson and Winternitz (1991) for the KP equation, it is sufficient to seek the similarity reduction in the special form

$$u(x, y, t) = \alpha(x, y, t) + \theta^2(y, t)w(\zeta(x, y, t), \tau(y, t)) \tag{3}$$

with

$$\zeta(x, y, t) = \theta(y, t)x + \varphi(y, t) \tag{4}$$

where $\alpha(x, y, t)$, $\tau(y, t)$, $\theta(y, t)$, and $\varphi(y, t)$ are to be determined. Substituting (3) and (4) into (1) yields

$$\begin{aligned} &\alpha_t + \alpha\alpha_x + \alpha_{xxx} + \alpha_{xyy} + (2\theta\theta_t + \alpha_x\theta^2)w \\ &\quad + [\alpha\theta^3 + 6\theta\theta_y^2 + 3\theta^2\theta_{yy} + \theta^2(x\theta_t + \varphi_t)]w_\zeta \\ &\quad + [6\theta^2\theta_y(x\theta_y + \varphi_y) + \theta^3(x\theta_{yy} + \varphi_{yy})]w_{\zeta\zeta} \\ &\quad + \theta^3[\theta^2 + (x\theta_y + \varphi_y)^2]w_{\zeta\zeta\zeta} + \theta^2\tau_t w_\tau + \theta^2(6\theta_y\tau_y + \theta\tau_{yy})w_{\zeta\tau} \\ &\quad + \theta^3\tau_y^2 w_{\zeta\tau\tau} + 2\theta^3(x\theta_y + \varphi_y)\tau_y w_{\zeta\zeta\tau} + \theta^5 w w_\zeta \\ &= 0 \end{aligned} \tag{5}$$

Equation (5) is a PDE of $w(\zeta, \tau)$ in two independent variables only; for the ratios of the coefficients of different partial derivatives and powers of $w(\zeta, \tau)$ being functions of ξ and τ, these conditions read

$$\alpha_t + \alpha\alpha_x + \alpha_{xxx} + \alpha_{xyy} = \theta^5\Gamma_1(\zeta, t) \tag{6}$$

$$2\theta\theta_t + \alpha_x\theta^2 = \theta^5\Gamma_2(\zeta, t) \tag{7}$$

$$\alpha\theta^3 + 6\theta\theta_y^2 + 3\theta^2\theta_{yy} + \theta^2(x\theta_t + \varphi_t) = \theta^5\Gamma_3(\zeta, \tau) \tag{8}$$

$$6\theta^2\theta_y(x\theta_y + \varphi_y) + \theta^3(x\theta_{yy} + \varphi_{yy} = \theta^5\Gamma_4(\zeta, \tau) \tag{9}$$

$$\theta^5 + \theta^3(x\theta_y + \varphi_y)^2 = \theta^5\Gamma_5(\zeta, \tau) \tag{10}$$

$$\theta^2\tau_t = \theta^5\Gamma_6(\zeta, \tau) \tag{11}$$

$$6\theta^2\theta_y\tau_y + \theta^3\tau_{yy} = \theta^5\Gamma_7(\zeta, \tau) \tag{12}$$

$$\theta^3\tau_y^2 = \theta^5\Gamma_8(\zeta, \tau) \tag{13}$$

$$2\theta^3(x\theta_y + \varphi_y)\tau_y = \theta^5\Gamma_9(\zeta, \tau) \tag{14}$$

where $\Gamma_i(\xi, \tau)$, $i = 1, 2, \ldots, 9$, and some functions of (ξ, τ) are to be determined.

Due to the freedom in choosing α, we may set $\Gamma_3(\zeta, \tau) = 0$ without loss of generality. From (8), we get

$$\alpha = -\frac{x\theta_t + \varphi_t}{\theta} - 6\left(\frac{\theta_y}{\theta}\right)^2 - \frac{3\theta_{yy}}{\theta} \tag{15}$$

To obtain the solutions of (6)–(14), there are two subcases to consider.

Case 2.1a. $\tau_y \neq 0$. By (13) and the freedom of τ, one can easily get

$$\tau = \int_0^y \theta(\eta, t)\, d\eta \tag{16}$$

and

$$\Gamma_8(\xi, \tau) = 1 \tag{17}$$

Substituting (16) into (12), we see that the freedom of $\theta(y, t)$ implies

$$\theta(y, t) = \theta(t) \tag{18}$$

and

$$\Gamma_7(\xi, \tau) = 0 \tag{19}$$

so

$$\tau = \theta(t)y + \psi(t) \tag{20}$$

with functions $\theta(t)$ and $\psi(t)$ to be determined.

By using (11) and (20), we obtain $\Gamma_6(\xi, \tau) = \tau$, and $\theta(t)$ and $\psi(t)$ satisfy

$$\theta_t = \theta^4 \tag{21}$$

$$\psi_t = \theta^3\psi \tag{22}$$

with general solution

$$\theta = [-3(t + t_0)]^{-1/3} \tag{23}$$

$$\psi = C_1[-3(t + t_0)]^{-1/3} \tag{24}$$

From (14), without loss of generality, we obtain

$$\varphi(y, t) = h(t) \tag{25}$$

where $h(t)$ satisfies

$$h_{tt} - 2\varphi^3 h_t - 2\varphi^6 h = 0 \tag{26}$$

It is easy to get the solutions of (26),

$$h(t) = C_2(t + t_0)^{2/3} \qquad \text{or} \qquad h(t) = C_2(t + t_0)^{-1/3} \tag{27}$$

where t_0 and C_2 are arbitrary constants, and (5) reduces to

$$\tau w_\tau + w_{\zeta\zeta\zeta} + w_{\zeta\tau\tau} + w w_\xi + w - 2\xi = 0 \tag{28}$$

Obviously $w = \zeta$ is a solution of (28).

Case 2.1b. $\tau_y = 0$ [i.e. $\tau = \tau(t)$]. Similar to case 2.1a, one can obtain the solutions of (6)–(14) as follows:

$$\theta = \theta_0, \qquad \alpha = 0, \qquad \varphi = p_0 y + d_0, \qquad \tau = \theta_0^3 t, \qquad \zeta = \theta_0 x + p_0 y + d_0 \tag{29}$$

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_7 = \Gamma_8 = \Gamma_9 = \Gamma_6 - 1 = \Gamma_5 - 1 - \frac{p_0^2}{\theta_0^2} = 0 \tag{30}$$

here $\theta_0 \neq 0$, and p_0 and d_0 are arbitrary constants. In this case, (5) becomes well-known KdV equation

$$w_\tau + w w_\zeta + \left[1 + \left(\frac{p_0}{\theta_0}\right)^2\right] w_{\zeta\zeta\zeta} = 0 \tag{31}$$

Case 2.2. $\zeta_x \neq 0$. With no loss of generality, we set $\zeta = y$, $\tau = \tau$, i.e., we seek a solution of the ZK equation in the special form

$$u = \alpha(x, y, t) + \beta(x, y, t) w(y, t) \tag{32}$$

Substituting (32) into equation (1) yields

$$\begin{aligned} &\alpha_t + \alpha\alpha_x + \alpha_{xxx} + \alpha_{xyy} + [\beta_t + (\alpha\beta)_x + \beta_{xxx} + \beta_{xyy}]w \\ &\quad + 2\beta_{xy} w_y + \beta_x w_{yy} + \beta w_t + \beta\beta_x w^2 = 0 \end{aligned} \tag{33}$$

This is a partial differential equation for $w(y, t)$; if the ratios of coefficients of different derivatives and powers of w are functions of t and y, the coefficient of w_{yy} and w_t yields

$$\beta_x = \beta\eta_1(y, t) \tag{34}$$

where $\eta_1(y, t)$ is to be determined; the integration gives

$$\beta = \exp(x\eta_1(y, t)) \tag{35}$$

Now consider the ratio of the coefficients of w^2 and w_t, i.e., $\eta_1(y, t)\exp(x\eta_1(y, t))$ is a function of y and t if and only if $\eta_1(y, t) = 0$. Hence $\beta = 1$, and equation (33) becomes

$$w_t + \alpha_x w + \alpha_t + \alpha\alpha_x + \alpha_{xxx} + \alpha_{xyy} = 0 \tag{36}$$

This is a partial differential equation for w with independent variables y and t provided that

$$\alpha_x = \eta_2(y, t) \tag{37}$$

$$\alpha_t + \alpha\alpha_x + \alpha_{xxx} + \alpha_{xyy} = \eta_3(y, t) \tag{38}$$

where $\eta_2(y, t)$ and $\eta_3(y, t)$ are to be determined. The integration of (37) yields

$$\alpha = \eta_2(y, t)x \tag{39}$$

Combining (39) and (38), we obtain

$$\eta_2 = \frac{1}{t + D_1(y)}, \qquad \eta_3 = \left(\frac{1}{t + D_1(y)}\right)_{yy} \tag{40}$$

with an arbitrary function $D_1(y)$. In this case (36) reduces to

$$w_t + \frac{1}{t + D_1(y)}w + \frac{2[D_1(y)]^2}{[t + D_1(y)]^3} - \frac{D_1''(y)}{[t + D_1(y)]^2} = 0 \tag{41}$$

with solution

$$w = \frac{2[D_1'(y)]^2}{[t + D_1(y)]^2} + \frac{D_1''(y)ln(t + D_1(y))}{t + D_1(y)} + \frac{D_2(y)}{t + D_t(y)} \tag{42}$$

where $D_2(y)$ also is an arbitrary function.

3. REDUCTIONS TO PDE BASED ON CONDITIONAL SYMMETRIES

In this section, by means of the nonclassical method due to Bluman and Cole (1969), we reduce the Zakharov–Kuznetsov equation to PDEs with two independent variables, i.e., we give the group explanation for the results of Section 2. Now we shall look for a transformation group leaving simultaneous solutions of two equations, namely (1) and

$$Q^{(2)} = \tau u_t + \zeta u_x + \eta u_y - \varphi = 0 \tag{43}$$

where τ, ζ, η, and φ are functions of t, x, y, and u to be determined.

The vector fields corresponding to the Lie group of local point transformations leaving the joint solution set of (1) and (43) invariant have the form

$$X = \tau \frac{\partial}{\partial t} + \zeta \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \varphi \frac{\partial}{\partial u} \tag{44}$$

We start by constructing the prolongation of X, i.e.,

$$\mathrm{Pr}^{(3)} X = X + \varphi^{t} \frac{\partial}{\partial u_t} + \varphi^{x} \frac{\partial}{\partial u_x} + \phi^{xxx} \frac{\partial}{\partial u_{xxx}} + \varphi^{xyy} \frac{\partial}{\partial u_{xyy}} \tag{45}$$

where the functions φ^t, φ^x, φ^{xxx}, and φ^{xyy} are functions of τ, ζ, η, φ, and their derivatives of t, x, and y; the prolongation is then used in the two equations and the resulting expressions are required to vanish on the solution set of the two equations, i.e.,

$$\mathrm{Pr}^{(3)} X(Q^{(1)})|_{Q^{(1)}=0, Q^{(2)}=0} = 0 \tag{46}$$

$$\mathrm{Pr}^{(1)} X(Q^{(2)})|_{Q^{(1)}=0, Q^{(2)}=0} = 0 \tag{47}$$

Equation (47) is satisfied automatically; (46) leads to a set of determining equations which must then be solved; there are several cases to consider.

Case 3.1. $\tau \neq 0$. Without loss of generality, suppose $\tau = 1$. MACSYMA provides determining equations; we obtain all constraint conditions, and the remaining nontrivial equations are

$$\zeta = \alpha(t)x + \beta(t), \qquad \eta = \delta(t)y + \Delta(t), \qquad \varphi = \gamma(t)u + \lambda(t) \tag{48}$$

$$\begin{gathered} \alpha = \delta = -\frac{\gamma}{2}, \qquad \alpha_t + 3\alpha^2 = 0, \qquad \Delta_t + 3\alpha\Delta = 0, \qquad \lambda_t + 3\alpha\lambda = 0 \\ \beta_t - \lambda + 3\alpha\beta = 0 \end{gathered} \tag{49}$$

with general solution

$$\begin{gathered} -\frac{\gamma}{2} = \delta = \alpha = \frac{1}{3(t+t_0)}, \qquad \Delta = \frac{D_3}{3(t+t_0)} \\ \lambda = \frac{D_4}{t+t_0}, \qquad \beta = \frac{D_5}{t+t_0} \end{gathered} \tag{50}$$

where D_3, D_4, and D_5 are the integration constants. So after multiplying by $t + t_0$, we find that the vector field X reads

$$\begin{aligned} X = {} & (t+t_0)\frac{\partial}{\partial t} + \left(\frac{1}{3}x + D_5\right)\frac{\partial}{\partial x} + \left(\frac{1}{3}y + D_3\right)\frac{\partial}{\partial y} \\ & + \left(-\frac{2}{3}u + D_4\right)\frac{\partial}{\partial u} \end{aligned} \tag{51}$$

By solving the characteristic equations, we can obtain the similarity variables; the similarity reduction we have derived is just case 2.1a of Section 2.

Case 3.2. $\tau = 0$, $\eta \neq 0$. With no loss of generality, we put $\eta = 1$; by solving the constraint conditions, we get

$$\xi = \beta_0, \qquad \varphi = \lambda_0 \tag{52}$$

and the vector field X reads

$$X = \beta_0 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \lambda \frac{\partial}{\partial u} \tag{53}$$

From this reduction, the reduction case 2.1b of Section 2 follows immediately.

Case 3.3. $\tau = 0$, $\eta = 1$. Without loss of generality, we set $\xi = 1$. Similar to cases 3.1 and 3.2, the constraint equations become simple

$$u = \lambda(t, x, y) \tag{54}$$

where λ satisfies

$$\lambda_x = 0, \qquad \lambda_t + \lambda^2 = 0 \tag{55}$$

with general solution

$$\lambda = \frac{1}{t + D_6(y)} \tag{56}$$

where $D_6(y)$ is an arbitrary function of y. Obviously, this reduction reduces to case 2.2 of Section 2.

4. REDUCTIONS TO ODE

In this section, we discuss the reductions of the ZK equation (1) to ordinary differential equations. Generally, there are two ways to do this, either by going directly from the ZK equation to ODEs in one step, or by considering reductions of the partial differential equations with two independent reductions in Section 2, which is a two-step procedure. Below we only give the results of applying the generalization of the direct method to get one-step reductions.

To obtain a one-step reduction of (1), we seek a reduction in the special form

$$u(x, y, t) = \alpha(x, y, t) + \beta(x, y, t)w(z(x, y, t)) \tag{57}$$

with α, β, w, and Z functions of indicated variables to be determined. Substituting (57) into (1), we get

$$\begin{aligned}&\alpha_t + \alpha\alpha_x + \alpha_{xxx} + \alpha_{xyy} + [\beta_t + (\alpha\beta)_x + \beta_{xxx} + \beta_{xyy}]w\\&\quad + (\beta z_t + \alpha\beta z_x + 3\beta_{xx} z_x + 3\beta_x z_{xx} + \beta z_{xxx} + 2\beta_{xy} z_y) + \beta_x z_{yy}\\&\quad + \beta_{yy} z_x + 2\beta_y z_{xy} + \beta z_{xyy} w_z + \beta\beta_x w^2 + \beta^2 z_x w w_z\\&\quad + [3\beta_x z_x^2 + 3\beta z_x z_{xx} + \beta_z z_y^2 + 2\beta_y z_x z_y + \beta(z_x z_y)_y] w_{zz}\\&\quad + \beta z_x (z_x^2 + z_y^2) w_{zzz} = 0 \end{aligned} \tag{58}$$

To determine the reductions, we omit the details of analysis, and only list the results.

Case 4.1. $z_x \neq 0$, $z_y \neq 0$, $z_{xy} \neq 0$. In this case, we have a generic reduction to an ordinary differential equation given by

$$u = -D_7 + \frac{1}{(y+y_0)^2} w(z(x, y, t))$$

where D_7 and y_0 are arbitrary constants, and

$$z(x, y, t) = -\frac{1}{y+y_0} x + \frac{D_8 + D_7 t}{y + y_0} \tag{59}$$

and for two arbitrary constants, w satisfies

$$\left(z^2 + \frac{7}{4}\right) w_{zz} - 10 z w_z + 10 w + \frac{w^2}{2} + D_8 = 0 \tag{60}$$

This reduction also can be obtained from case 2.1a by a one-step reduction procedure.

Case 4.2. $z_x \neq 0$, $z_y \neq 0$, $z_{xy} = 0$. The reduction in this case is given by

$$u = \alpha(x, y, t) + \beta(t) w(z(x, y, t)) \tag{61}$$

where α, β, and z are determined by

$$\alpha = -(x+y)\frac{\theta_t}{\theta} - \frac{\sigma_t}{\theta}, \qquad \beta = \theta^2(t), \qquad z = (x+y)\theta(t) + \sigma(t) \tag{62}$$

with $\theta(t)$ and $\sigma(t)$ satisfying

$$\theta_t = \Gamma_0 \theta^4 \tag{63}$$

$$\theta\sigma_{tt} - 2\theta_t \sigma_t = 2\theta^6 (\Gamma_0^2 \sigma + \Gamma_1) \tag{64}$$

where Γ_0 and Γ_1 are two arbitrary constants and w satisfies

$$\theta^5 (2w''' + ww') + \theta\theta_t w + \alpha_t + \alpha\alpha_x = 0 \tag{65}$$

Case 4.3. $z_x \neq 0$, $z_y = 0$. In this case, the reduction can be obtained from case 4.2 by using x instead of $x + y$, but with w satisfying

$$\theta^5(w''' + ww') + \theta\theta_t w + \alpha_t + \alpha\alpha_x = 0 \tag{66}$$

This reduction is just the same as the reduction of the KdV equation (Clarkson and Kruskal, 1989, p. 2211).

Case 4.4. $z_x = 0$, $z_y = 0$, i.e., $z = z(t) = t$. In this case (58) reduces to

$$\alpha_t + \alpha\alpha_x + \alpha_{xxx} + \alpha_{xyy} + [\beta_t + (\alpha\beta)_x + \beta_{xxx} + \beta_{xyy}]w + \beta w_t + \beta\beta_x w^2 = 0 \tag{67}$$

Hence, the ZK equation (1) has the reduction

$$u = -C_1(y) + (x + C_1(y)t + C_2(y))w(t) \tag{68}$$

with two arbitrary functions $C_1(y)$ and $C_2(y)$, and w satisfies

$$w_t + w^2 = 0$$

with solution

$$w = \frac{1}{t + t_0} \tag{69}$$

From the point of view of the symmetry group, we can easily give the interpretation for these reductions, but omit it here.

5. SUMMARY AND DISCUSSION

Originally, Ablowitz *et al.* (1980) conjectured that a nonlinear partial differential equation is integrable if all its exact reductions to ODEs have the Painlevé property, but this approach poses the obvious operational difficulty of finding all the exact reductions. Clarkson and Kruskal (1989) introduced a valid method, the direct method, which has been successfully applied to obtain new symmetry reductions and exact solutions for several physically significant PDEs (Clarkson and Kruskal, 1989; Lou, 1990, 1992; Lou and Ruan, 1993; Clarkson and Winternitz, 1991; Nucci and Clarkson, 1992).

Although the integrability of the ZK equation has been studied by several authors (Infeld and Frycz, 1989; Shivamoggi, 1989, 1990; Shivamoggi and Rollins, 1991), we do not know whether the ZK equation is integrable. In this paper, to provide some information about the integrability of the ZK equation, we have derived some new reductions to PDEs and ODEs by the direct method. Following the method of Weiss *et al.* (1983), we find that the reduction equation (28) possesses the conditional Painlevé property and equation (31) is the well-known KdV equation and thus has the Painlevé property. According to Clarkson and Kruskal (1989), the reduction equations (65) and (66) possess the Painlevé property. It is worthwhile to notice that the solution of (33) contains an algebraic branch point and an algorithmic branch point; this usually exists in an integrable system.

In Section 3 we gave the symmetry group interpretation for the results of Section 2, by the nonclassical method due to Bluman and Cole. Generally, the nonclassical method is more general than the direct method (Nucci and Clarkson, 1992). It is possible that we have not found all the reductions of the ZK equation, which we leave to further study.

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